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ANTI-SELF-DUAL HERMITIAN METRICS AND PAINLEVÉ III

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0. Introduction

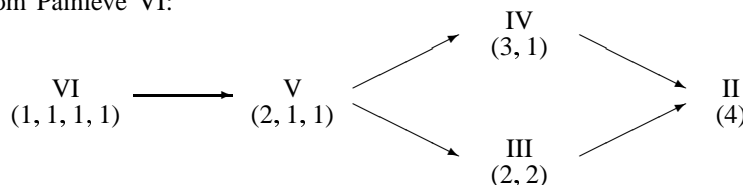
The aim of this paper is to study the $SU(2)$ -invariant anti-self-dual metrics which is specified by the solutions of Painlevé III. We study not only the diagonal metrics, but also the non-diagonal metrics.

Hitchin [6] shows that the $SU(2)$ -invariant anti-self-dual metric is generically specified by a solution of Painlevé VI with two complex parameters.

Painlevé VI is shown to be a deformation equation for a linear problem

$$\left(\frac{d}{dz} - B_1\right) \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = 0,$$

where B_1 has four simple poles on \mathbb{CP}^1 [7]. And Painlevé V, IV, III, II are degenerated from Painlevé VI:



This is the confluence diagram of poles of B_1 , where the Roman numerals represent the types of the Painlevé equation, and the parenthesized numbers represent the orders of poles of B_1 . For example, Painlevé III is shown to be a deformation equation for a linear problem with two double poles.

Hitchin used the twistor correspondence [1, 11] to associate the anti-self-dual equation and the Painlevé equation. On the twistor space, the lifted action of $SU(2)$ determines a pre-homogeneous action of $SU(2)$, and it determines an isomonodromic family of connections on \mathbb{CP}^1 , and then we obtain the Painlevé equation.

Due to the reality condition of the twistor space, the poles of B_1 makes two antipodal pairs. Therefore, the configuration of poles becomes the type of Painlevé III or VI. Generically, the anti-self-dual metric is specified by a solution of Painlevé VI.

In this framework, Hitchin [6] classified the diagonal anti-self-dual metrics, and Dancer [5] shows that the diagonal scalar-flat Kähler metric is specified by a solution of Painlevé III with a parameter $(0, 4, 4, -4)$, where *diagonal* metric is in the shape of (1) in Section 1. Since the anti-self-dual Einstein metrics are diagonal, the classifi-

cation for diagonal metrics enough serves Hitchin's purpose. However, generically, the $SU(2)$ -invariant metric is in the shape of (4) in Section 2. In this case, Hitchin shows that the metric is generically specified by a solution of Painlevé VI, but he does not go into detail. In this paper, we study not only the diagonal metrics but also the non-diagonal metrics.

We show that the $SU(2)$ -invariant anti-self-dual equations reduce to the following Painlevé equations:

(a) A family of Painlevé VI

$$\frac{d^2 q}{dx^2} = \frac{1}{2} \left(\frac{1}{q} + \frac{1}{q-1} + \frac{1}{q-x} \right) \left(\frac{dq}{dx} \right)^2 - \left(\frac{1}{x} + \frac{1}{x-1} + \frac{1}{q-x} \right) \frac{dq}{dx} + \frac{q(q-1)(q-x)}{x^2(x-1)^2} \left\{ \alpha + \beta \frac{x}{q^2} + \gamma \frac{x-1}{(q-1)^2} + \delta \frac{x(x-1)}{(q-x)^2} \right\}$$

with two complex parameters,

$$(\alpha, \beta, \gamma, \delta) = \left(\frac{1}{2}(\theta_0 - 1)^2, \frac{1}{2}\bar{\theta}_0^2, -\frac{1}{2}\theta_1^2, \frac{1}{2}(1 + \bar{\theta}_1^2) \right).$$

If the metric is in the form (15), then $\theta_0 = \theta_1$ or $\theta_0, \theta_1 \in \mathbb{R}$.

(b) A family of Painlevé III

$$\frac{d^2 q}{dx^2} = \frac{1}{q} \left(\frac{dq}{dx} \right)^2 - \frac{1}{x} \frac{dq}{dx} + \frac{1}{x} (\alpha q^2 + \beta) + \gamma q^3 + \frac{\delta}{q}.$$

with one complex parameter,

$$(\alpha, \beta, \gamma, \delta) = (4\theta, 4(1 + \bar{\theta}), 4, -4).$$

If the metric is in the form (15), then $\theta \in \mathbb{R}$.

The case (b) is a generalization of Dancer's result [5].

Generically, the $SU(2)$ -invariant anti-self-dual metric is specified by a solution of Painlevé VI with a parameter above. The metric is specified by a solution of Painlevé III, if and only if there exists an $SU(2)$ -invariant hermitian structure. With an appropriate conformal rescaling, the hermitian metric turns into a scalar-flat Kähler metric.

1. The diagonal anti-self-dual equations

In this section, we review the anti-self-dual equations on the $SU(2)$ -invariant diagonal metrics.

The $SU(2)$ -invariant diagonal metric is represented in the following form:

$$(1) \quad g = w_1 w_2 w_3 dt^2 + \frac{w_2 w_3}{w_1} \sigma_1^2 + \frac{w_3 w_1}{w_2} \sigma_2^2 + \frac{w_1 w_2}{w_3} \sigma_3^2.$$

w_1, w_2 and w_3 are functions of t , and $\sigma_1, \sigma_2, \sigma_3$ are left invariant one-forms on each $SU(2)$ -orbit satisfying

$$(2) \quad d\sigma_1 = \sigma_2 \wedge \sigma_3, \quad d\sigma_2 = \sigma_3 \wedge \sigma_1, \quad d\sigma_3 = \sigma_1 \wedge \sigma_2.$$

Tod [12] showed that the (scalar-flat) anti-self-dual equations on the $SU(2)$ -invariant diagonal metric are given by the following system:

$$(3) \quad \begin{aligned} \dot{w}_1 &= -w_2 w_3 + w_1 (\alpha_2 + \alpha_3), \\ \dot{w}_2 &= -w_3 w_1 + w_2 (\alpha_3 + \alpha_1), \\ \dot{w}_3 &= -w_1 w_2 + w_3 (\alpha_1 + \alpha_2), \\ \dot{\alpha}_1 &= -\alpha_2 \alpha_3 + \alpha_1 (\alpha_2 + \alpha_3), \\ \dot{\alpha}_2 &= -\alpha_3 \alpha_1 + \alpha_2 (\alpha_3 + \alpha_1), \\ \dot{\alpha}_3 &= -\alpha_1 \alpha_2 + \alpha_3 (\alpha_1 + \alpha_2), \end{aligned}$$

where $\alpha_1, \alpha_2, \alpha_3$ are auxiliary functions and the dots denote differentiation with respect to t . The anti-self-dual equation (3) has a first integral

$$k = \frac{\alpha_1(w_2^2 - w_3^2) + \alpha_2(w_3^2 - w_1^2) + \alpha_3(w_1^2 - w_2^2)}{8(\alpha_1 - \alpha_2)(\alpha_2 - \alpha_3)(\alpha_3 - \alpha_1)}.$$

Furthermore, if we set

$$\begin{aligned} x &= \frac{\alpha_2 - \alpha_1}{\alpha_2 - \alpha_3}, \\ q &= \frac{w_2(\alpha_1 - \alpha_2)(w_2(w_1^2 - w_3^2) + 2\sqrt{2k}w_1w_3(\alpha_1 - \alpha_3))}{w_1^2(w_2^2 - w_3^2)\alpha_1 + w_2^2(w_3^2 - w_1^2)\alpha_2 + w_3^2(w_1^2 - w_2^2)\alpha_3}, \end{aligned}$$

then the system (3) generically reduces to a family of Painlevé VI with a special parameter

$$(\alpha, \beta, \gamma, \delta) = \left(\frac{(\sqrt{2k} - 1)^2}{2}, k, -k, \frac{1 + 2k}{2} \right).$$

2. The non-diagonal anti-self-dual equations

We can express an $SU(2)$ -invariant metric in the form

$$(4) \quad g = f(\tau) d\tau^2 + \sum_{l,m=1}^3 h_{lm}(\tau) \sigma_l \sigma_m,$$

Using the Killing form, we can diagonalize the metric g on each $SU(2)$ -orbit. Then we can express the metric as follows:

$$g = (abc)^2 dt^2 + a^2 d\delta_1^2 + b^2 d\delta_2^2 + c^2 d\delta_3^2,$$

where $t = t(\tau)$, $a = a(t)$, $b = b(t)$, $c = c(t)$ and

$$\begin{pmatrix} \hat{\sigma}_1 \\ \hat{\sigma}_2 \\ \hat{\sigma}_3 \end{pmatrix} = R(t) \begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \end{pmatrix},$$

where $R(t)$ is an $SO(3)$ -valued function.

Since $\dot{R}R^{-1} \in \mathfrak{so}(3)$, we obtain

$$\begin{aligned} d \begin{pmatrix} \hat{\sigma}_1 \\ \hat{\sigma}_2 \\ \hat{\sigma}_3 \end{pmatrix} &= R(t) \begin{pmatrix} \sigma_2 \wedge \sigma_3 \\ \sigma_3 \wedge \sigma_1 \\ \sigma_2 \wedge \sigma_2 \end{pmatrix} + \dot{R} dt \wedge \begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \end{pmatrix} \\ &= \begin{pmatrix} \hat{\sigma}_2 \wedge \hat{\sigma}_3 \\ \hat{\sigma}_3 \wedge \hat{\sigma}_1 \\ \hat{\sigma}_1 \wedge \hat{\sigma}_2 \end{pmatrix} + \begin{pmatrix} 0 & \xi_3 & -\xi_2 \\ -\xi_3 & 0 & \xi_1 \\ \xi_2 & -\xi_1 & 0 \end{pmatrix} dt \wedge \begin{pmatrix} \hat{\sigma}_1 \\ \hat{\sigma}_2 \\ \hat{\sigma}_3 \end{pmatrix}, \end{aligned}$$

for some $\xi_1 = \xi_1(t)$, $\xi_2 = \xi_2(t)$, $\xi_3 = \xi_3(t)$.

If $\xi_1 = 0$, $\xi_2 = 0$, $\xi_3 = 0$, then the matrix (h_{lm}) can be chosen to be diagonal for all τ , and then we call that g has a diagonal form.

In the following, we mainly study the non-diagonal metrics.

To compute the curvature tensor, we choose a basis for \bigwedge^2

$$\{\Omega_1^+, \Omega_2^+, \Omega_3^+, \Omega_1^-, \Omega_2^-, \Omega_3^-\},$$

where

$$\begin{aligned} \Omega_1^+ &= a^2 bc dt \wedge \hat{\sigma}_1 + bc \hat{\sigma}_2 \wedge \hat{\sigma}_3, \\ \Omega_2^+ &= ab^2 c dt \wedge \hat{\sigma}_2 + ca \hat{\sigma}_3 \wedge \hat{\sigma}_1, \\ \Omega_3^+ &= abc^2 dt \wedge \hat{\sigma}_3 + ab \hat{\sigma}_1 \wedge \hat{\sigma}_2, \\ \Omega_1^- &= a^2 bc dt \wedge \hat{\sigma}_1 - bc \hat{\sigma}_2 \wedge \hat{\sigma}_3, \\ \Omega_2^- &= ab^2 c dt \wedge \hat{\sigma}_2 - ca \hat{\sigma}_3 \wedge \hat{\sigma}_1, \\ \Omega_3^- &= abc^2 dt \wedge \hat{\sigma}_3 - ab \hat{\sigma}_1 \wedge \hat{\sigma}_2. \end{aligned}$$

With respect to this frame, the curvature tensor has the following block form [3]:

$$\begin{pmatrix} A & B \\ {}^t B & D \end{pmatrix},$$

where $s = 4 \text{ trace } D$ is the scalar curvature, $W^+ = A - (1/12)s$ and $W^- = D - (1/12)s$ are the self-dual and anti-self-dual parts of the Weyl tensor, and B is the trace free parts of Ricci tensor.

We set $w_1 = bc$, $w_2 = ca$, $w_3 = ab$ and determine auxiliary functions α_1 , α_2 , α_3 by

$$(5) \quad \begin{aligned} \dot{w}_1 &= -w_2w_3 + w_1(\alpha_2 + \alpha_3), \\ \dot{w}_2 &= -w_3w_1 + w_2(\alpha_3 + \alpha_1), \\ \dot{w}_3 &= -w_1w_2 + w_3(\alpha_1 + \alpha_2). \end{aligned}$$

Calculating the condition $A = 0$, we obtain the following theorem.

Theorem 2.1. *The metric is anti-self-dual with vanishing scalar curvature, if and only if α_1 , α_2 , α_3 and ξ_1 , ξ_2 , ξ_3 satisfy the following equations:*

$$(6) \quad \begin{aligned} \dot{\alpha}_1 &= -\alpha_2\alpha_3 + \alpha_1(\alpha_2 + \alpha_3) + \frac{1}{4}(w_2^2 - w_3^2)^2 \left(\frac{\xi_1}{w_2w_3} \right)^2 \\ &\quad + \frac{1}{4}(w_3^2 - w_1^2)(3w_1^2 + w_3^2) \left(\frac{\xi_2}{w_3w_1} \right)^2 \\ &\quad + \frac{1}{4}(w_2^2 - w_1^2)(3w_1^2 + w_2^2) \left(\frac{\xi_3}{w_1w_2} \right)^2, \\ \dot{\alpha}_2 &= -\alpha_3\alpha_1 + \alpha_2(\alpha_3 + \alpha_1) + \frac{1}{4}(w_3^2 - w_1^2)^2 \left(\frac{\xi_2}{w_3w_1} \right)^2 \\ &\quad + \frac{1}{4}(w_1^2 - w_2^2)(3w_2^2 + w_1^2) \left(\frac{\xi_3}{w_1w_2} \right)^2 \\ &\quad + \frac{1}{4}(w_3^2 - w_2^2)(3w_2^2 + w_3^2) \left(\frac{\xi_1}{w_2w_3} \right)^2, \\ \dot{\alpha}_3 &= -\alpha_1\alpha_2 + \alpha_3(\alpha_1 + \alpha_2) + \frac{1}{4}(w_1^2 - w_2^2)^2 \left(\frac{\xi_3}{w_1w_2} \right)^2 \\ &\quad + \frac{1}{4}(w_2^2 - w_3^2)(3w_3^2 + w_2^2) \left(\frac{\xi_1}{w_2w_3} \right)^2 \\ &\quad + \frac{1}{4}(w_1^2 - w_3^2)(3w_3^2 + w_1^2) \left(\frac{\xi_2}{w_3w_1} \right)^2, \end{aligned}$$

and

$$\begin{aligned}
 (w_2^2 - w_3^2) \frac{d}{dt} \left(\frac{\xi_1}{w_2 w_3} \right) &= \frac{\xi_2}{w_3 w_1} \frac{\xi_3}{w_1 w_2} (-2w_2^2 w_3^2 + w_3^2 w_1^2 + w_1^2 w_2^2) \\
 &\quad + \frac{\xi_1}{w_2 w_3} (\alpha_2 w_2^2 - \alpha_3 w_3^2 + 3\alpha_2 w_3^2 - 3\alpha_3 w_2^2), \\
 (w_3^2 - w_1^2) \frac{d}{dt} \left(\frac{\xi_2}{w_3 w_1} \right) &= \frac{\xi_3}{w_1 w_2} \frac{\xi_1}{w_2 w_3} (-2w_3^2 w_1^2 + w_1^2 w_2^2 + w_2^2 w_3^2) \\
 &\quad + \frac{\xi_2}{w_3 w_1} (\alpha_3 w_3^2 - \alpha_1 w_1^2 + 3\alpha_3 w_1^2 - 3\alpha_1 w_3^2), \\
 (w_1^2 - w_2^2) \frac{d}{dt} \left(\frac{\xi_3}{w_1 w_2} \right) &= \frac{\xi_1}{w_2 w_3} \frac{\xi_2}{w_3 w_1} (-2w_1^2 w_2^2 + w_2^2 w_3^2 + w_3^2 w_1^2) \\
 &\quad + \frac{\xi_3}{w_1 w_2} (\alpha_1 w_1^2 - \alpha_2 w_2^2 + 3\alpha_1 w_2^2 - 3\alpha_2 w_1^2).
 \end{aligned}
 \tag{7}$$

REMARK 2.2. If we take a conformal rescaling g to $F(t)g$, then t turns into s that satisfies $ds/dt = 1/F$, and w_1, w_2, w_3 turn into Fw_1, Fw_2, Fw_3 , and ξ_1, ξ_2, ξ_3 turn into $F\xi_1, F\xi_2, F\xi_3$ respectively. And then $\alpha_1, \alpha_2, \alpha_3$ turn into

$$\tilde{\alpha}_1 = \frac{1}{2} \frac{dF}{dt} + F\alpha_1, \quad \tilde{\alpha}_2 = \frac{1}{2} \frac{dF}{dt} + F\alpha_2, \quad \tilde{\alpha}_3 = \frac{1}{2} \frac{dF}{dt} + F\alpha_3.$$

The equations (5), (6), (7) are invariant under a conformal rescaling g to Fg , if $2F\dot{F}^2 = \ddot{F}^2$.

REMARK 2.3. By the equation (5), (6), (7), we obtain $-2w_1^2 w_2^2 + w_2^2 w_3^2 + w_3^2 w_1^2 \neq 0$. Therefore, if $\xi_3 \equiv 0$, then $\xi_1 \equiv 0$ or $\xi_2 \equiv 0$. In the same way, if $\xi_1 \equiv 0$, then $\xi_2 \equiv 0$ or $\xi_3 \equiv 0$, and if $\xi_2 \equiv 0$, then $\xi_3 \equiv 0$ or $\xi_1 \equiv 0$.

REMARK 2.4. If $\xi_1 = 0, \xi_2 = 0, \xi_3 = 0$, then the equation (5), (6), (7) reduces to a sixth-order system (3) given by Tod [12]. Furthermore, if $\alpha_1 = w_1, \alpha_2 = w_2, \alpha_3 = w_3$, then (5), (6), (7) reduce to a third-order system, which determines Atiyah-Hitchin family [1]. And if $\alpha_1 = 0, \alpha_2 = 0, \alpha_3 = 0$, then (5), (6), (7) reduce to a third-order system, which determines BGPP family [4].

REMARK 2.5. If $w_2 = w_3$, then we can set $\xi_1 = 0, \xi_2 = 0, \xi_3 = 0$ by taking another frame. This is also in the diagonal case. Therefore we assume $(w_2 - w_3)(w_3 - w_1)(w_1 - w_2) \neq 0$.

3. The isomonodromic deformations

Let (M, g) be an oriented Riemannian four manifold. We define a manifold Z to be a unit sphere bundle in the bundle of anti-self-dual two-forms, and let $\pi: Z \rightarrow M$

denote the projection. Each point z in the fiber over $\pi(z)$ defines a complex structure on the tangent space $T_{\pi(z)}M$, compatible with the metric and its orientation.

Using the Levi-Civita connection, we can split the tangent space $T_z Z$ into horizontal and vertical spaces, and the projection π identifies the horizontal space with $T_{\pi(z)}M$. This space has a complex structure defined by z and the vertical space is the tangent space of the fiber $S^2 \cong \mathbb{CP}^1$ which has its natural complex structure.

The almost complex structure on Z is integrable, if and only if the metric is anti-self-dual [2, 11]. In this situation Z is called the twistor space of (M, g) and the fibers are called the real twistor lines.

The almost complex structure on Z can be determined by the following $(1, 0)$ -forms:

$$\begin{aligned} \Theta_1 &= z(e^1 + \sqrt{-1}e^2) - (e^0 + \sqrt{-1}e^3), \\ \Theta_2 &= z(e^0 - \sqrt{-1}e^3) + (e^1 - \sqrt{-1}e^2), \\ (8) \quad \Theta_3 &= dz + \frac{1}{2}z^2(\omega_1^0 - \omega_3^2 + \sqrt{-1}(\omega_2^0 - \omega_1^3)) \\ &\quad - \sqrt{-1}z(\omega_3^0 - \omega_2^1) + \frac{1}{2}(\omega_1^0 - \omega_3^2 - \sqrt{-1}(\omega_2^0 - \omega_1^3)), \end{aligned}$$

where $\{e^0, e^1, e^2, e^3\}$ is an orthonormal frame, and ω_j^i are the connection forms determined by $de^i + \omega_j^i \wedge e^j = 0$ and $\omega_j^i + \omega_i^j = 0$. (M, g) is anti-self-dual, if and only if the Pfaffian system (or the distribution defined by the following system)

$$(9) \quad \Theta_1 = 0, \quad \Theta_2 = 0, \quad \Theta_3 = 0$$

is integrable, that is to say

$$(10) \quad d\Theta_1 \equiv 0, \quad d\Theta_2 \equiv 0, \quad d\Theta_3 \equiv 0 \quad (\text{mod } \Theta_1, \Theta_2, \Theta_3).$$

REMARK 3.1. The Pfaffian system (9) is invariant under $z \mapsto (z + \sqrt{-1})/(z - \sqrt{-1})$ and permutation $1 \mapsto 2, 2 \mapsto 3, 3 \mapsto 1$ of suffixes of e^i and ω_j^i .

Theorem 3.2. *The Pfaffian system (9) is invariant under the conjugate action and $z \mapsto -1/\bar{z}$ [2].*

If the metric is $SU(2)$ invariant, we obtain

$$(11) \quad \begin{pmatrix} \Theta_1 \\ \Theta_2 \\ \Theta_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} dz + \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} dt + A \begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \end{pmatrix},$$

where $v_1 = v_1(z, t)$, $v_2 = v_2(z, t)$, $v_3 = v_3(z, t)$; $A = (a_{ij}(z, t))_{i,j=1,2,3}$.

If $\det A \equiv 0$, then the metric turns to be diagonal, and the metric is in the BGPP family [4].

If $\det A \neq 0$, then we obtain

$$\begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \end{pmatrix} \equiv -A^{-1} \left(\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} dz + \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} dt \right), \quad (\text{mod } \Theta_1, \Theta_2, \Theta_3).$$

If we set

$$(12) \quad \begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix} = -A^{-1} \left(\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} dz + \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} dt \right),$$

then

$$(13) \quad d \begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix} \equiv \begin{pmatrix} s_2 \wedge s_3 \\ s_3 \wedge s_1 \\ s_1 \wedge s_2 \end{pmatrix}, \quad (\text{mod } \Theta_1, \Theta_2, \Theta_3).$$

Since s_1, s_2, s_3 are one-forms on (z, t) -plane, the congruency equation (13) turns to be a plain equation:

$$d \begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix} = \begin{pmatrix} s_2 \wedge s_3 \\ s_3 \wedge s_1 \\ s_1 \wedge s_2 \end{pmatrix}.$$

By Theorem 3.2, s_1, s_2, s_3 are invariant under the conjugate action and $z \mapsto -1/\bar{z}$.

If we set

$$\begin{aligned} \Sigma &= \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{-1} s_2 & -s_1 + \sqrt{-1} s_3 \\ s_1 + \sqrt{-1} s_3 & -\sqrt{-1} s_2 \end{pmatrix} \\ &=: -B_1 dz - B_2 dt, \end{aligned}$$

then

$$d\Sigma + \Sigma \wedge \Sigma = 0.$$

This is the isomonodromic condition for the following linear problem [7]

$$(14) \quad \left(\frac{d}{dz} - B_1 \right) \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = 0.$$

Lemma 3.3. *The components of B_1 are rational functions of z ,*

$$B_1 = \frac{F(z)}{G(z)},$$

where $F(z)$ is degree 2 and $G(z)$ is degree 4. We must have $B_1 \mapsto -^t B_1$ under the conjugate action and $z \mapsto -1/\bar{z}$.

Proof. Since s_1, s_2, s_3 are invariant under the conjugate action and $z \mapsto -1/\bar{z}$, we obtain $\Sigma \mapsto -^t \Sigma$, and then $B_1 \mapsto -^t B_1$.

If we set

$$\begin{pmatrix} \hat{s}_1 \\ \hat{s}_2 \\ \hat{s}_3 \end{pmatrix} = R(t) \begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix},$$

then we have

$$\begin{pmatrix} \hat{s}_1 \\ \hat{s}_2 \\ \hat{s}_3 \end{pmatrix} \equiv \begin{pmatrix} \hat{\sigma}_1 \\ \hat{\sigma}_2 \\ \hat{\sigma}_3 \end{pmatrix} \pmod{\Theta_1, \Theta_2, \Theta_3}.$$

By a straightforward calculation, we obtain

$$\begin{aligned} \hat{s}_1 \left(\frac{\partial}{\partial z} \right) &\equiv \frac{2(1+z^2)w_1}{G(z)}, \\ \hat{s}_2 \left(\frac{\partial}{\partial z} \right) &\equiv \frac{2\sqrt{-1}(-1+z^2)w_2}{G(z)}, \\ \hat{s}_3 \left(\frac{\partial}{\partial z} \right) &\equiv \frac{-4\sqrt{-1}zw_3}{G(z)}, \end{aligned}$$

where

$$\begin{aligned} G(z) = z^4 &\left((\alpha_1 - \alpha_2) - \sqrt{-1} \frac{w_1^2 - w_2^2}{w_1 w_2} \xi_3 \right) - 2z^3 \left(\frac{w_2^2 - w_3^2}{w_2 w_3} \xi_1 - \sqrt{-1} \frac{w_3^2 - w_1^2}{w_3 w_1} \xi_2 \right) \\ &+ 2z^2 (\alpha_1 + \alpha_2 - 2\alpha_3) + 2z \left(\frac{w_2^2 - w_3^2}{w_2 w_3} \xi_1 + \sqrt{-1} \frac{w_3^2 - w_1^2}{w_3 w_1} \xi_2 \right) \\ &+ \left((\alpha_1 - \alpha_2) + \sqrt{-1} \frac{w_1^2 - w_2^2}{w_1 w_2} \xi_3 \right). \end{aligned}$$

Since $R(t)$ is independent of z , the components of B_1 are rational functions of z ,

$$B_1 = \frac{F(z)}{G(z)},$$

where $F(z)$ is degree 2 and $G(z)$ is degree 4. □

For this lemma, generically B_1 has four simple poles. In this case, the deformation equation of (14) is Painlevé VI.

Theorem 3.4. *The $SU(2)$ -invariant anti-self-dual metric is generically specified by the solution of Painlevé VI.*

The idea of Hitchin [6] is that the lifted action of $SU(2)$ on the twistor space Z gives a homomorphism of vector bundles $\alpha: Z \times \mathfrak{su}(2)^{\mathbb{C}} \rightarrow TZ$, and the inverse of α gives a flat meromorphic $SL(2, \mathbb{C})$ -connection, which determines isomonodromic deformations. Since one-forms $\Theta_1, \Theta_2, \Theta_3$ on Z can be considered as infinitesimal variations, we can identify Σ with α^{-1} .

By Lemma 3.3, the poles of B_1 make antipodal pairs $\zeta_0, -1/\bar{\zeta}_0$, and $\zeta_1, -1/\bar{\zeta}_1$ on \mathbb{CP}^1 . Therefore we obtain two types of configuration of poles of B_1 . In each case, we can calculate the local exponents at singularities. These local exponents corresponding to the parameter of the Painlevé equation (see [8]).

(a) B_1 has four simple poles $\zeta_0, -1/\bar{\zeta}_0, \zeta_1, -1/\bar{\zeta}_1$ on \mathbb{CP}^1 .

$$B_1 = \frac{A_0}{z - \zeta_0} + \frac{-{}^t\bar{A}_0}{z + 1/\bar{\zeta}_0} + \frac{A_1}{z - \zeta_1} + \frac{-{}^t\bar{A}_1}{z + 1/\bar{\zeta}_1}.$$

The deformation equation is Painlevé VI with a parameter,

$$(\alpha, \beta, \gamma, \delta) = \left(\frac{1}{2}(\theta_0 - 1)^2, \frac{1}{2}\bar{\theta}_0^2, -\frac{1}{2}\theta_1^2, \frac{1}{2}(1 + \bar{\theta}_1^2) \right),$$

where $\theta_0^2 = 2 \operatorname{tr} A_0^2$, $\theta_1^2 = 2 \operatorname{tr} A_1^2$.

(b) B_1 has two double poles $\zeta, -1/\bar{\zeta}$ on \mathbb{CP}^1 .

$$B_1 = \frac{A_2}{(z - \zeta)^2} + \frac{\sqrt{-1}C}{z - \zeta} + \frac{-\sqrt{-1}C}{z + 1/\bar{\zeta}} + \frac{-{}^t\bar{A}_2/\bar{\zeta}^2}{(z + 1/\bar{\zeta})^2},$$

where $C = -{}^t\bar{C}$. The deformation equation is Painlevé III with a parameter,

$$(\alpha, \beta, \gamma, \delta) = (4\theta, 4(1 + \bar{\theta}), 4, -4),$$

where $\theta^2 = 2(\operatorname{tr}(A_2C))^2/\operatorname{tr} C^2$.

Theorem 3.5. *The anti-self-dual equations reduce to the following Painlevé equations:*

(a) *A family of Painlevé VI with two complex parameters,*

$$(\alpha, \beta, \gamma, \delta) = \left(\frac{1}{2}(\theta_0 - 1)^2, \frac{1}{2}\bar{\theta}_0^2, -\frac{1}{2}\theta_1^2, \frac{1}{2}(1 + \bar{\theta}_1^2) \right).$$

(b) *A family of Painlevé III with one complex parameter,*

$$(\alpha, \beta, \gamma, \delta) = (4\theta, 4(1 + \bar{\theta}), 4, -4).$$

REMARK 3.6. It is known that the anti-self-dual equations reduce to Painlevé VI with the parameter as above ([9], [6]). Dancer [5] shows the diagonal scalar-flat Kähler metric is specified by a solution of Painlevé III with a parameter $(\alpha, \beta, \gamma, \delta) = (0, 4, 4, -4)$. Now, Theorem 3.5 (b) is a generalization of Dancer's result.

By Remark 2.3, if $\xi_1 \xi_2 \xi_3 = 0$, then at least two of ξ_1, ξ_2, ξ_3 must be zero. From now on this section, we assume $\xi_2 = \xi_3 = 0$, and then we obtain the metric in the form:

$$(15) \quad g = f(\tau) d\tau + h_{11}(\tau) \sigma_1^2 + h_{22}(\tau) \sigma_2^2 + h_{23}(\tau) \sigma_2 \sigma_3 + h_{33}(\tau) \sigma_3^2.$$

Therefore, there exists an isometric action

$$(16) \quad (\sigma_1, \sigma_2, \sigma_3) \mapsto (\sigma_1, -\sigma_2, -\sigma_3),$$

which preserves each orbit. Since

$$\begin{pmatrix} \Theta_1 \\ \Theta_2 \\ \Theta_3 \end{pmatrix} \mapsto \begin{pmatrix} \bar{\Theta}_1 \\ \bar{\Theta}_2 \\ \bar{\Theta}_3 \end{pmatrix}$$

under the action (16) and $z \mapsto \bar{z}$, then we obtain

$$\begin{pmatrix} \Theta_1 \\ \Theta_2 \\ \Theta_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} dz + \overline{\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \bigg|_{z \mapsto \bar{z}}} dt + \overline{A|_{z \mapsto \bar{z}}} \begin{pmatrix} \sigma_1 \\ -\sigma_2 \\ -\sigma_3 \end{pmatrix}.$$

Therefore

$$\begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix} \bigg|_{z \mapsto \bar{z}} = \begin{pmatrix} \bar{s}_1 \\ -\bar{s}_2 \\ -\bar{s}_3 \end{pmatrix},$$

and then we obtain $B_1|_{z \mapsto \bar{z}} = \bar{B}_1$. Therefore we obtain the following:

(a) If B_1 has four simple poles, then

$$\begin{aligned} B_1 &= \frac{A_0}{z - \zeta_0} + \frac{-{}^t \bar{A}_0}{z + 1/\bar{\zeta}_0} + \frac{A_1}{z - \zeta_1} + \frac{-{}^t \bar{A}_1}{z + 1/\bar{\zeta}_1} \\ &= \frac{\bar{A}_0}{z - \bar{\zeta}_0} + \frac{-{}^t A_0}{z + 1/\zeta_0} + \frac{\bar{A}_1}{z - \bar{\zeta}_1} + \frac{-{}^t A_1}{z + 1/\zeta_1}. \end{aligned}$$

If $\zeta_0 = \bar{\zeta}_0$ or $-1/\zeta_0$, then $\theta_0^2 = 2 \operatorname{tr} A_0^2$ and $\theta_1^2 = 2 \operatorname{tr} A_1^2$ must be real numbers. If $\zeta_0 = \bar{\zeta}_1$ or $-1/\zeta_1$, then $\theta_0^2 = 2 \operatorname{tr} A_0^2$ and $\theta_1^2 = 2 \operatorname{tr} A_1^2$ must coincide.

(b) If B_1 has two double poles, then

$$\begin{aligned} B_1 &= \frac{A_2}{(z - \zeta)^2} + \frac{\sqrt{-1} C}{z - \zeta} + \frac{-\sqrt{-1} C}{z + 1/\bar{\zeta}} + \frac{-{}^t\bar{A}_2/\bar{\zeta}^2}{(z + 1/\bar{\zeta})^2} \\ &= \frac{\bar{A}_2}{(z - \bar{\zeta})^2} + \frac{\sqrt{-1} \bar{C}}{z - \bar{\zeta}} + \frac{-\sqrt{-1} \bar{C}}{z + 1/\zeta} + \frac{-{}^t A_2/\zeta^2}{(z + 1/\zeta)^2}, \end{aligned}$$

where $C = -{}^t\bar{C}$. If $\zeta = \bar{\zeta}$, then $\theta^2 = 2(\text{tr}(A_2 C))^2 / \text{tr} A_2^2$ must be a real number. If $\zeta = -1/\bar{\zeta}$, then $\theta^2 = 2(\text{tr}(A_2 C))^2 / \text{tr} A_2^2 = 0$.

Theorem 3.7. *If $\xi_1 \xi_2 \xi_3 = 0$, then the anti-self-dual equations reduce to the following Painlevé equations:*

(a) *A family of Painlevé VI with two real parameters,*

$$(\alpha, \beta, \gamma, \delta) = \left(\frac{1}{2}(\theta_0 - 1)^2, \frac{1}{2}\theta_0^2, -\frac{1}{2}\theta_1^2, \frac{1}{2}(1 + \theta_1^2) \right),$$

or one complex parameter,

$$(\alpha, \beta, \gamma, \delta) = \left(\frac{1}{2}(\theta - 1)^2, \frac{1}{2}\bar{\theta}^2, -\frac{1}{2}\theta^2, \frac{1}{2}(1 + \bar{\theta}^2) \right).$$

(b) *A family of Painlevé III with one real parameter,*

$$(\alpha, \beta, \gamma, \delta) = (4\theta, 4(1 + \theta), 4, -4).$$

4. Hermitian structure

In this section, we study the geometric meaning of the $SU(2)$ -invariant anti-self-dual metric specified by the solutions of Painlevé III. Painlevé III is the deformation equation of

$$\left(\frac{d}{dz} - B_1 \right) \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = 0,$$

where B_1 has two double poles. By a direct calculation, we obtain the following lemma.

Lemma 4.1. *The poles of B_1 are determined by the following equation:*

$$\begin{aligned} (17) \quad z^4 & \left((\alpha_1 - \alpha_2) - \sqrt{-1} X_3 \right) - 2z^3 \left(X_1 - \sqrt{-1} X_2 \right) \\ & + 2z^2 (\alpha_1 + \alpha_2 - 2\alpha_3) + 2z \left(X_1 + \sqrt{-1} X_2 \right) \\ & + \left((\alpha_1 - \alpha_2) + \sqrt{-1} X_3 \right) = 0, \end{aligned}$$

where

$$X_1 = \frac{w_2^2 - w_3^2}{w_2 w_3} \xi_1, \quad X_2 = \frac{w_3^2 - w_1^2}{w_3 w_1} \xi_2, \quad X_3 = \frac{w_1^2 - w_2^2}{w_1 w_2} \xi_3.$$

Since the equation (17) is preserved by $z \mapsto -1/\bar{z}$ and the conjugate action, if the equation (17) has a solution $z = \zeta$ of order two, then $z = -1/\bar{\zeta}$ is also a solution of order two.

Lemma 4.2. *Let g be a non-diagonal $SU(2)$ -invariant metric. Then B_1 has two double poles, if and only if there exists a function $f(t)$ satisfying*

$$(18) \quad \begin{aligned} X_1^2 &= 4(f - \alpha_2)(f - \alpha_3), \\ X_2^2 &= 4(f - \alpha_3)(f - \alpha_1), \\ X_3^2 &= 4(f - \alpha_1)(f - \alpha_2). \end{aligned}$$

And then the anti-self-dual equation reduce to (5), (6) and $\dot{f} = f^2$.

Proof. If $X_1 = X_2 = X_3 = 0$, then the discriminant of (17) is

$$16(\alpha_1 - \alpha_2)^2(\alpha_2 - \alpha_3)^2(\alpha_3 - \alpha_1)^2.$$

Therefore, if

$$(\alpha_1 - \alpha_2)(\alpha_2 - \alpha_3)(\alpha_3 - \alpha_1) = 0,$$

then B_1 has two double poles. This case is in the form of (18) by $f = \alpha_1 = \alpha_2$ or $f = \alpha_2 = \alpha_3$ or $f = \alpha_3 = \alpha_1$. By the equation (5), (6), (7), we obtain $\dot{f} = f^2$. If $f = 0$, then we obtain the diagonal scalar-flat-Kähler metric given by Pedersen and Poon [10].

If $X_1 X_2 X_3 = 0$, then, from Remark 2.3, at least two of X_1, X_2, X_3 must be zero. Assume that $X_1 \neq 0$ and $X_2 = X_3 = 0$. Then the discriminant of (17) is $(X_1^2 + (\alpha_2 - \alpha_3)^2)(X_1^2 - 4(\alpha_1 - \alpha_2)(\alpha_1 - \alpha_3))^2$. Therefore, the equation

$$X_1^2 = 4(\alpha_1 - \alpha_2)(\alpha_1 - \alpha_3)$$

is the condition that B_1 has two double poles. This is (18) where $f = \alpha_1$. In this case, we obtain the double poles on

$$(19) \quad \zeta = \frac{\sqrt{\alpha_3 - \alpha_1} \pm \sqrt{-1} \sqrt{\alpha_2 - \alpha_1}}{\sqrt{\alpha_2 + \alpha_3 + 2\alpha_1}}.$$

X_1, X_2, X_3 satisfy the equation (5), (6), (7), if and only if $\alpha_1 = \alpha_1^2$.

If $X_1 X_2 X_3 \neq 0$, the discriminant of (17) is too complicated to calculate. Therefore, we attack by an other way. We obtain (17) in the following form:

$$(20) \quad \bar{a} z^4 - \bar{b} z^3 + c z^2 + b z + a = 0,$$

where a, b are complex coefficients and c is a real coefficient. By a linear fractional transformation

$$(21) \quad z \mapsto \frac{(b - |b|) \zeta - b + |b|}{(-\bar{b} + |b|) \zeta - \bar{b} + |b|},$$

the equation (17) turns into the following form:

$$(22) \quad \zeta^4 - \bar{b}_0 \zeta^3 + c_0 \zeta^2 + b_0 \zeta + 1 = 0,$$

where b_0 is a complex coefficient and c_0 is a real coefficients. Since (21) preserves the antipodal pairs on \mathbb{CP}^1 , if $\zeta = \zeta_0$ is a solution of (22) of order two, then $\zeta = -1/\bar{\zeta}_0$ is also a solution of order two. Therefore

$$(23) \quad \zeta^4 - \bar{b}_0 \zeta^3 + c_0 \zeta^2 + b_0 \zeta + 1 = (\zeta - \zeta_0)^2 \left(\zeta + \frac{1}{\bar{\zeta}_0} \right)^2,$$

and then $\zeta_0 = \pm \bar{\zeta}_0$, which implies ζ_0 is real or pure-imaginary. Therefore $b_0 = 2\zeta(-1 + \zeta\bar{\zeta})/\bar{\zeta}^2$ must be real or pure-imaginary. By a direct calculation, we obtain the following. The real part of b_0 vanishes, if and only if

$$X_2^4 (X_1^2 + X_2^2)^2 = 0,$$

which never occurs. The imaginary part of b_0 vanishes, if and only if

$$X_2^4 ((X_1^2 - X_2^2) X_3 - 2X_1 X_2 (\alpha_1 - \alpha_2)) = 0.$$

Therefore,

$$(24) \quad (X_1^2 - X_2^2) X_3 = 2X_1 X_2 (\alpha_1 - \alpha_2),$$

if and only if B_1 has two double poles. By the linear transformation $z \mapsto (z + \sqrt{-1})/(z - \sqrt{-1})$, the suffixes of X_i and α_i on (17) are replaced cyclically. Therefore, if B_1 has two double poles, then the following must be also satisfied:

$$(25) \quad (X_2^2 - X_3^2) X_1 = 2X_2 X_3 (\alpha_2 - \alpha_3),$$

$$(26) \quad (X_3^2 - X_1^2) X_2 = 2X_3 X_1 (\alpha_3 - \alpha_1).$$

By (24), (25) and (26), X_1, X_2, X_3 must satisfy (18) with an auxiliary function f . Actually, if X_1, X_2, X_3 satisfy (18), then (17) has two solutions of order two:

$$(27) \quad \zeta = \frac{X_1 X_2 \pm \sqrt{X_2^2 X_3^2 + X_3^2 X_1^2 + X_1^2 X_2^2}}{X_3(X_1 - \sqrt{-1} X_2)}.$$

In this case, X_1, X_2, X_3 satisfy the equation (5), (6), (7), if and only if $\dot{f} = f^2$. \square

Therefore, we obtain the following theorem.

Theorem 4.3. *The $SU(2)$ -invariant anti-self-dual metric is specified by the solution of Painlevé III, if and only if X_1, X_2, X_3 satisfy (18) and $\dot{f} = f^2$.*

If we restrict $\Theta_1|_{z=\zeta(t)}$ and $\Theta_2|_{z=\zeta(t)}$ for some $z = \zeta(t)$, then we obtain $(1, 0)$ -forms on M , which determine an $SU(2)$ -invariant almost complex structure on M .

Theorem 4.4. *Let g be an $SU(2)$ -invariant anti-self-dual scalar-flat metric. There exists an $SU(2)$ -invariant hermitian structure (g, I) if and only if B_1 has double poles.*

Proof. Let $G(z)$ be the left hand side of (17). Then $G(z)$ is the denominator of B_1 . We obtain

$$\Theta_3 \equiv dz + H_0 dt + H_1 \hat{\sigma}_1 \pmod{\Theta_1, \Theta_2},$$

where $H_1 = 0$ is equivalent with $G(z) = 0$, and $dz + H_0 dt = 0$ is equivalent with $dG = 0$. Therefore, the almost complex structure determined by $\{\Theta_1|_{z=\zeta(t)}, \Theta_2|_{z=\zeta(t)}\}$ is integrable, if and only if $G(z)$ admits a multiple zero on $z = \zeta(t)$. \square

Theorem 4.5. *The hermitian structure (g, I) determined on Theorem 4.4 is Kähler, if and only if*

$$(28) \quad X_1^2 = 4\alpha_2\alpha_3, \quad X_2^2 = 4\alpha_3\alpha_1, \quad X_3^2 = 4\alpha_1\alpha_2.$$

Proof. If $X_1 X_2 X_3 \neq 0$, the Kähler form is determined by (27) as

$$\begin{aligned} \Omega &= \frac{X_2 X_3}{\sqrt{X_2^2 X_3^2 + X_3^2 X_1^2 + X_1^2 X_2^2}} \Omega_1^+ \\ &\quad + \frac{X_3 X_1}{\sqrt{X_2^2 X_3^2 + X_3^2 X_1^2 + X_1^2 X_2^2}} \Omega_2^+ \end{aligned}$$

$$+ \frac{X_1 X_2}{\sqrt{X_2^2 X_3^2 + X_3^2 X_1^2 + X_1^2 X_2^2}} \Omega_3^+.$$

By the equations (5), (6), (7) and $\dot{f} = f^2$, we obtain

$$\begin{aligned} d\Omega = & \frac{2f w_1 X_2 X_3}{\sqrt{X_2^2 X_3^2 + X_3^2 X_1^2 + X_1^2 X_2^2}} dt \wedge \hat{\sigma}_2 \wedge \hat{\sigma}_3 \\ & + \frac{2f w_2 X_3 X_1}{\sqrt{X_2^2 X_3^2 + X_3^2 X_1^2 + X_1^2 X_2^2}} dt \wedge \hat{\sigma}_3 \wedge \hat{\sigma}_1 \\ & + \frac{2f w_3 X_1 X_2}{\sqrt{X_2^2 X_3^2 + X_3^2 X_1^2 + X_1^2 X_2^2}} dt \wedge \hat{\sigma}_1 \wedge \hat{\sigma}_2. \end{aligned}$$

Since $w_1 w_2 w_3 \neq 0$ and $X_1 X_2 X_3 \neq 0$, we obtain $d\Omega = 0$, if and only if $f = 0$.

If $X_1 X_2 X_3 = 0$, then f must be α_1 , α_2 or α_3 . Suppose that $f = \alpha_1$, then we obtain $X_1^2 = 4(\alpha_2 - \alpha_1)(\alpha_3 - \alpha_1)$, $X_2 = 0$, $X_3 = 0$. The Kähler form is determined by (19) as

$$(29) \quad \Omega = \frac{\sqrt{\alpha_2 - \alpha_1}}{\sqrt{\alpha_2 + \alpha_3 - 2\alpha_1}} \Omega_2^+ + \frac{\sqrt{\alpha_3 - \alpha_1}}{\sqrt{\alpha_2 + \alpha_3 - 2\alpha_1}} \Omega_3^+.$$

Then we obtain

$$(30) \quad d\Omega = \frac{2w_2 \alpha_1 \sqrt{\alpha_2 - \alpha_1}}{\sqrt{\alpha_2 + \alpha_3 - 2\alpha_1}} dt \wedge \hat{\sigma}_3 \wedge \hat{\sigma}_1 + \frac{2w_3 \alpha_1 \sqrt{\alpha_3 - \alpha_1}}{\sqrt{\alpha_2 + \alpha_3 - 2\alpha_1}} dt \wedge \hat{\sigma}_1 \wedge \hat{\sigma}_2.$$

If the metric is non-diagonal, then $X_1^2 = 4(\alpha_2 - \alpha_1)(\alpha_3 - \alpha_1) \neq 0$. Therefore, we obtain $d\Omega = 0$, if and only if $\alpha_1 = 0$. \square

By a conformal rescaling $g \mapsto Fg$ where F satisfies $(1/2)(dF/dt) = f$, we can eliminate f of lemma 4.2 (see Remark 2.2).

Theorem 4.6. *An $SU(2)$ -invariant anti-self-dual metric is specified by a solution of Painlevé III, if and only if the metric is conformally equivalent with a scalar-flat Kähler metric.*

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References

- [1] M.F. Atiyah and N.J. Hitchin: *Low energy scattering of non-Abelian monopoles*, Phys. Lett. A **107** (1985), 21–25.
- [2] M.F. Atiyah, N.J. Hitchin and I.M. Singer: *Self-duality in four-dimensional Riemannian geometry*, Proc. Roy. Soc. London Ser. A **362** (1978) 425–461.
- [3] A.L. Besse: *Einstein Manifolds*, Springer-Verlag, Berlin-Heidelberg, 1987.
- [4] V.A. Belinski, G.W. Gibbons, D.W. Page and C.N. Pope: *Asymptotically Euclidean Bianchi IX metrics in quantum gravity*, Phys. Lett. B **76** (1978), 433–435.
- [5] A.S. Dancer: *Scalar-flat Kähler metrics with $SU(2)$ symmetry*, J. reine. angew. Math. **479** (1996), 99–120.
- [6] N.J. Hitchin: *Twistor spaces, Einstein metrics and isomonodromic deformations*, J. Differential Geom. **42** (1995), 30–112.
- [7] M. Jimbo, T. Miwa and K. Ueno: *Monodromy Preserving Deformation of Linear Ordinary Differential Equations with Rational Coefficients. I*, Physica **2D** (1981), 306–352.
- [8] M. Jimbo and T. Miwa: *Monodromy Preserving Deformation of Linear Ordinary Differential Equations with Rational Coefficients. II*, Physica **2D** (1981), 407–448.
- [9] R. Maszczyk, L.J. Mason and N.M.J. Woodhouse: *Self-dual Bianchi metrics and the Painlevé transcendents*, Classical Quantum Gravity **11** (1994), 65–71.
- [10] H. Pedersen and Y.S. Poon: *Kähler surfaces with zero scalar curvature*, Classical Quantum Gravity **7** (1990), 1707–1719.
- [11] R. Penrose: *Nonlinear gravitons and curved twistor theory*, Gen. Rel. Grav. **7** (1976), 31–52.
- [12] K.P. Tod: *Self-dual Einstein metrics from the Painlevé VI equation*, Phys. Lett. A **190** (1994), 221–224.

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